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Math 131 - Spring 2024 - Common Final Exam, version A

Print name:______________________________________

Print instructor's name:______________________________________

Directions:

- This exam has 14 questions worth a total of 100 points.
- Fill in your name and instructor's name above.
- Show your work. Answers (even correct ones) without the corresponding work will receive no credit.
- You may use a calculator which cannot connect to the internet. The use of any notes or electronic devices other than a calculator is prohibited.

Good luck!

Question:	1	2	3	4	5	6	7	8
Points:	6	5	13	11	5	6	6	8
Score:								

Question:	9	10	11	12	13	14	Total
Points:	10	8	5	4	5	8	100
Score:							

1. After putting money into an investment fund for *t* months, your balance is *B* dollars, where $B = g(t)$. Give a one-sentence interpretation of each statement below. You must include units in your interpretation.

   a. (2 points) $g(20) = 1345$

      Example answer : "At 20 months, the balance in the fund is $1345."
      1 pt for correct input and output
      1 pt for correct units throughout

   b. (2 points) $g'(20) = -12$

      Example answer: "At 20 months, the account is losing $12 per month."
      1 pt for correct input and output
      1 pt for correct units throughout

   c. (2 points) $\int_0^{20} g'(t)dt = 155$

      Example answer : "Over the first 20 months, the total change in the fund's balance is $155."
      1 pt for correct input and output
      1 pt for correct units throughout

2. (5 points) The revenue (in thousands of dollars) generated by a company's service is found using the function $R(q) = \sqrt{e^q + 3q}$ where *q* is the number of times the service is provided to a client. Using calculus, estimate the additional revenue generated by increasing *q* one unit at $q = 5$. Provide units and round your answer to two decimal places.

   SOLUTION: $\boldsymbol{R'(5) = 5.92 \text{ thousands of } \$ \; per \; unit}$
   1 pt for evidence of chain rule
   2 pts for correct R'(q)
   1 pt for correct R'(5)
   1 pt for correct units

3. Let $f(x) = x^3 - 3x^2 + 1$ have domain $(-\infty, \infty)$.

a. (2 points) Find $f'(x)$.

SOLUTION: $\boldsymbol{f'(x) = 3x^2 - 6x}$ **(2 pts)**

b. (2 points) Find the x-coordinates of all critical points of $f(x)$.

SOLUTIONS: Crit pts at x = 0 and x = 2 (1 pt each)

c. (3 points) Using either the first or second-derivative test, find and classify all critical points of $f(x)$ as a local maximum, local minimum, or neither. You must show your work.

SOLUTIONS: Local min at x = 2, local max at x = 0
1 pt for evidence of either derivative test used.
1 pt for each correct classification.

d. (3 points) Does $f(x)$ have a global maximum? Does $f(x)$ have a global minimum? Explain your answers using a complete sentence for each.

SOLUTIONS: There is neither a global max or min
1 pt for "no" for each question, or "Neither exists".
1 pt each for justifications that use the end behavior or range in the explanation.

e. (3 points) Use calculus to find the x-coordinates of any inflection points. Justify your answer in a complete sentence

SOLUTION: Inflection point at x = 1.
1 pt for setting f''(x) = 0
1 pt for possible inflection point at x = 1
1 pt for a check for concavity change on either side of x = 1

4. The table below gives several values of a continuous, invertible function $f(x)$. Assume that the domain of both $f(x)$ and $f'(x)$ is the interval $(-\infty, \infty)$.

x	0	4	8	12	16	20	24	28	32
$f(x)$	-9	-5	-1	4	5	8.5	11	13.5	16

a. (3 points) Evaluate each of the following. **(1 pt for each correct answer.)**

   i. $f(f(32))$ = **5**

   ii. $f^{-1}(11)$ = **24**

   iii. $f^{-1}(f(4) + 4)$ = **8**

b. (2 points) Compute the *average rate of change* of f on the interval $12 \leq x \leq 24$. Show your work.

   SOLUTION: Average rate of change = $\frac{7}{12}$
   1 pt for some difference quotient evaluated
   1 pt for correct dq and solution

c. (2 points) Estimate $f'(20)$. You must show your work.

   SOLUTION: $\boldsymbol{f'(20) \approx \frac{3}{4} = 0.75}$
   1 pt for some difference quotient evaluated
   1 pt for a correct dq average and solution

d. (2 points) Suppose $f'(24) = 0.6$. $Estimate\ f(23)$. Show your work.

   SOLUTION: $\boldsymbol{f(23) \approx 10.4}$
   2 pts for correct use of given f' and answer

e. (2 points) If $f''(24) = -0.3$, do you expect your estimate in (d) to be an *overestimate* or an *underestimate*? Explain.

   SOLUTION: Overestimate
   1 pt for "Overestimate"
   1 pt for correct statement about the fact that f'(x) is decreasing at x=24

5. (5 points) Evaluate the following definite integral <u>exactly</u> using the Fundamental Theorem of Calculus. Show your work. A calculator solution will earn no credit.

$$\int_0^2 (4x + 2e^x)dx$$

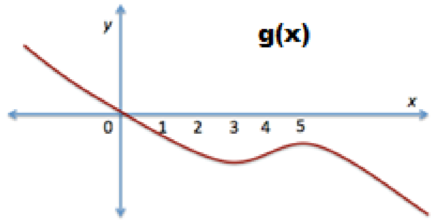
SOLUTION: $\mathbf{6 + 2e^2}$
2 pts for correct antiderivative
2 pts for correct use of x = 0 and x = 2 in the FToC
1 pt for exact answer

6. (6 points) Let $\frac{dy}{dt} = 4 + 3\sin(t)$. Find a formula for *y* in terms of *t* with the initial condition *y* = 7 when *t* = 0. Show your work.

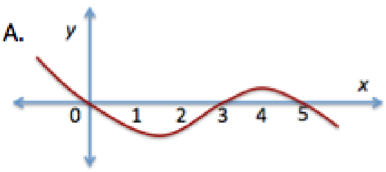
SOLUTION: $\boldsymbol{y = 4t - 3\cos(t) + 10}$
2 pts for correct antiderivative
1 pt for including the constant of integration
1 pt for the correct use of t = 0 and y = 7
2 pts for correct equation for y

7.

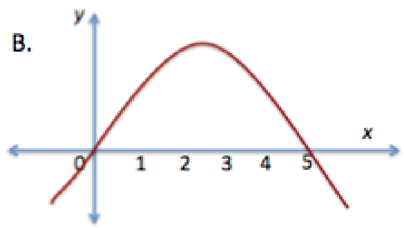
a. (3 points) The graph of $y = g(x)$ is shown here. Which one of the graphs below it could be the graph of its derivative, $g'(x)$? **Choose one and write the letter in the box below the graphs.**



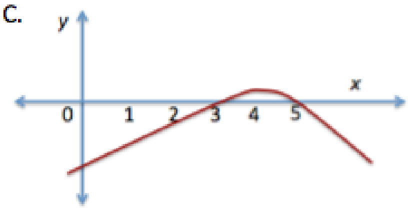
A.



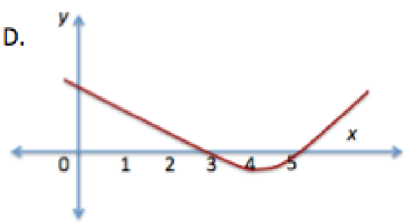
B.



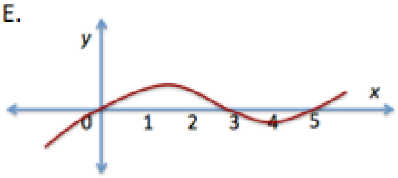
C.



D.



E.



The graph of $g'(x)$ is **C** **(3 pts for correct choice)**

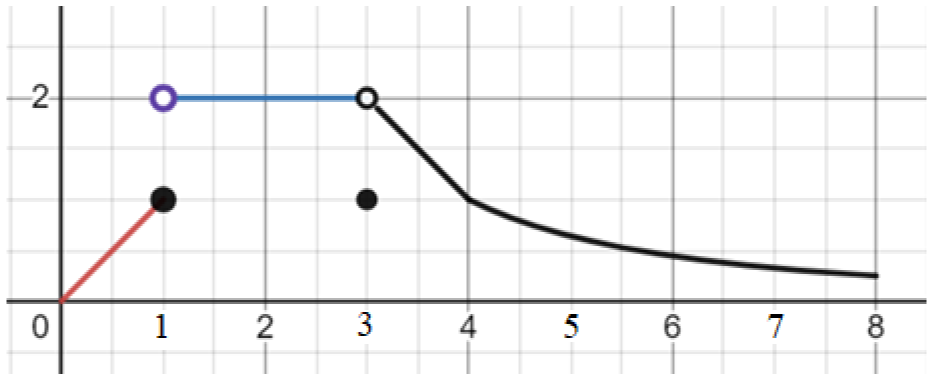
b. (3 points) Provide an explanation for your choice in part (a). State clearly why your choice is the only possible answer.

1 pt for any correct statement relating tangent line slopes to values of g'(x)

1 pt for at least one correct connection between tangent line slopes of g(x) and the chosen answer

1 pt for for a correct statement about the tangent line slopes of g(x) and C on the interval $(-\infty, 3)$

8. Consider the graph of $y = f(x)$ given below with domain $[0, 8]$.



a. (2 point) At which of the following points does $\lim_{x \to p} f(x)$ exist?

Circle all correct answers or write NONE.

p = 1 (p = 2) (p = 3) (p = 4) (p = 5) (p = 6) (p = 7)

2 pts for completely correct answer

b. (2 point) What is the value of $\lim_{x \to 1^-} f(x)$?

Circle all correct answers.

0 (1) 2 3 DNE

2 pts for selecting only the value 1

c. (2 point) At which of the following points is f continuous?

Circle all correct answers or write NONE.

p = 1 (p = 2) p = 3 (p = 4) (p = 5) (p = 6) (p = 7)

2 pts for completely correct answer

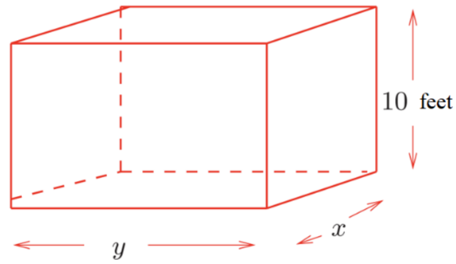
d. (2 points) Find $f'(3.5)$. Show your work.

SOLUTION: $\boldsymbol{f'(3.5) = -1}$

1 pt for attempt at finding the slope of the line at x = 3.5

1 pt for correct value

9. Pandora has decided to build a display tank to house her carnivorous plants. She wants five sides to be made of glass and the top to be open. She also demands that the height be 10 feet.



Let the width be *x* (in feet), length be *y* (in feet), and the volume be *V*.

a. (2points) Express the volume of the box in terms of *x* and *y*.

SOLUTION: $V = 10xy$

1 pt for use of box volume formula

1 pt for correct expression

b. (2 points) Write a formula for the total surface area, *A*, of the glass needed to build this box as a function of *x* and *y*.

SOLUTION: $A = 20x + 20y + xy$

1 pt for any summation of the rectangular side areas

1 pt for correct expression

c. (3 points) Pandora decides she wants the total volume to be 1000 cubic feet. Express the surface area, *A*, as a function of a single variable.

SOLUTION: $A = 20x + \frac{2000}{x} + 100 \quad or \quad A = 20y + \frac{2000}{y} + 100$

1 pt for an attempt to set V = 1000 and solve for one variable

1 pt for $y = \frac{100}{x} \quad or \quad x = \frac{100}{y}$

1 pt for correct substitution back into expression for A

d. (3 points) Given that the glass needed is very expensive, using calculus, find *x* and *y* that minimize the surface area needed to make the box. Show all work.

SOLUTIONS: x = 10 feet and y = 10 feet

1 pt for correct derivative $A'(q)$

1 pt for $A'(q) = 0$

1 pt for correct values of x and y.

10. Suppose that we wish to calculate the area under the curve $h(x) = \ln x$ over the interval [2, 6].

a. (3 points) Approximate this area using a left-endpoint sum (i.e. Riemann left sum) with $n = 4$. Express your answer to the nearest tenth. Show your work. *(*An answer produced using only a calculator will earn no credit.)

SOLUTION: 4.8 (2 pts)

1 pt for using only the left hand values of h(x) and $\Delta x = 1$

b. (2 points) Is your answer from part (a) an overestimate or an underestimate of the true value of the area? Explain your answer in a complete sentence.

SOLUTION: Underestimate (1 pt)

1 pt for correct justification

Example justification: "For an increasing function like h(x), each left hand value will be the lowest value for each rectangle's interval."

c. (3 points) Given that $H(x) = x \ln x - x$ is an antiderivative of $h(x) = \ln x$, use the Fundamental Theorem of Calculus to find the exact area beneath the curve $h(x) = \ln x$ over the interval [2, 6].

SOLUTION: Area = $6 \ln(6) - 2 \ln(2) - 4$

1 pt for any evidence of the FToC

1 pt for correct use of the FToC

1 pt for exact valued answer

11. (5 points) Given the function *q(x)* below, use calculus to find a *q'(0)*. Simplify your answer fully. You must show all work for full credit.

$$q(x) = \frac{2 \sin(x)}{x^2 + 3}$$

SOLUTION: $q'(0) = \frac{2}{3}$

1 pt for some evidence of the quotient rule

1 pt for both correct derivatives (f' and g')

2 pts for correct q'(x)

1 pt for correct value for q'(0)

12. (4 points) Let $A(x) = x^x$

Use the limit definition of the derivative to write an explicit expression for $A'(4)$. Your answer should not involve the letter *A*. Do not attempt to evaluate or simplify the limit. Please write your final answer in the answer box provided below.

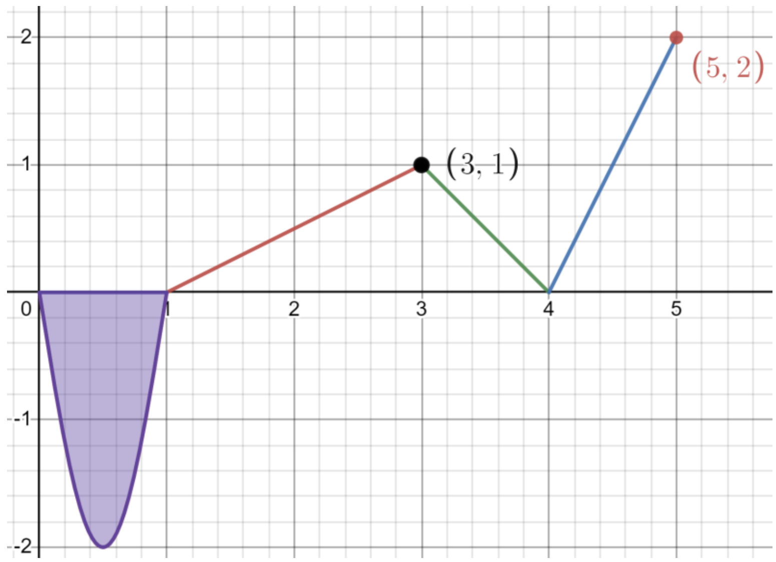
1 pt for some difference quotient with a limit as h→ $\mathbf{0}$
1 pt for correct substitution of 4 + h
2 pts for correct limit and dq for A'(4) below

SOLUTION: $\boldsymbol{A'(4) = \lim_{h\to 0}\left(\frac{(4+h)^{4+h}-256}{h}\right)}$

13. (5 points) The number of people being vaccinated in a small city is steadily increasing. Let $P(t) = 5t \arctan(t)$ represent the total number vaccinated (in thousands of people) over *t* weeks. What is the rate of change of vaccinated people at the 8-week mark? You must use calculus and show all work. Include units and round your final answer to 2 decimal places.

SOLUTION: $\boldsymbol{P'(t) = 7.85\ thousands\ of\ people\ per\ week}$
1 pt for evidence of the product rule
2 pts for correct expression for P'(t)
1 pt for P'(8) = 7.85
1 pt for correct units

14. Below is the graph of a function $y = f(x)$.



Assume that the area of the shaded region is 1.6.

a. (4 points) Find $\int_0^5 f(x)dx$. Show your work.

SOLUTION: 0.9

1 pt for using -1.6 not +1.6

1 pt for evidence of area calculations

2 pts for correct answer

b. (4 points) Find $\int_1^3 (3 - 2f(x))dx$. Show your work.

SOLUTION: 4

1 pt for splitting the integral correctly

1 pt for finding $\int_1^3 3\, dx = 6$

1 pt for finding area of triangle on [1 , 3]

1 pt for correct answer

Elementary Tools from Algebra and Geometry

Quadratic Formula: $ax^2 + bx + c = 0 \implies x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Rectangle Area = base $\times$ height

Surface area of a 3D shape = sum of all side areas

Volume of a rectangular box = length $\times$ width $\times$ height.

Five derivative rules for operations on functions.

Constant Multiples: $\dfrac{d}{dx}\Big(cf(x)\Big) = cf'(x)$

Sums & Differences: $\dfrac{d}{dx}\Big(f(x) \pm g(x)\Big) = f'(x) \pm g'(x)$

Product Rule: $\dfrac{d}{dx}\Big(f(x) \cdot g(x)\Big) = f'(x)g(x) + f(x)g'(x)$

Quotient Rule: $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

Chain Rule: $\dfrac{d}{dx}\Big(f(g(x))\Big) = f'(g(x)) \cdot g'(x)$

Ten derivative rules for functions

Derivative of a Constant: $\dfrac{d}{dx}(c) = 0$

The Power Rule: $\dfrac{d}{dx}\Big(x^n\Big) = nx^{n-1}$

Exponential Functions: General Case: $\dfrac{d}{dx}\Big(a^x\Big) = a^x \cdot \ln(a)$ Special Case: $\dfrac{d}{dx}\Big(e^x\Big) = e^x$

Three Trigonometric Rules:

$\dfrac{d}{dx}\Big(\sin(x)\Big) = \cos(x)$

$\dfrac{d}{dx}\Big(\cos(x)\Big) = -\sin(x)$

$\dfrac{d}{dx}\Big(\tan(x)\Big) = \sec^2(x) = \dfrac{1}{\cos^2(x)}$

Three Inverse Function Rules:

$\dfrac{d}{dx}\Big(\ln(x)\Big) = \dfrac{1}{x}$

$\dfrac{d}{dx}\Big(\arctan(x)\Big) = \dfrac{1}{1+x^2}$

$\dfrac{d}{dx}\Big(\arcsin(x)\Big) = \dfrac{1}{\sqrt{1-x^2}}$

General Antiderivative Rules

If k is a constant $\displaystyle\int k\,dx = kx + C$

$\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$, when $n \neq -1$

$\displaystyle\int a^x\,dx = \frac{a^x}{\ln(a)} + C$

$\displaystyle\int e^x\,dx = e^x + C$

$\displaystyle\int \cos(x)\,dx = \sin(x) + C$

$\displaystyle\int \sin(x)\,dx = -\cos(x) + C$

$\displaystyle\int \sec^2(x)\,dx = \tan(x) + C$

$\displaystyle\int \frac{1}{x}\,dx = \ln(|x|) + C$

$\displaystyle\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$

$\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$